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Lattices of sound tubes with harmonically related eigenfrequencies

J.-P. Dalmont and J. Kergomard

Laboratoire d'Acoustique de l'Université du Maine, URA 1101 CNRS, Av. Olivier Messiaen, B.P. 535, 72017 Le Mans Cedex, France

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Abstract. — The only continuous acoustic horns (in plane wave approximation) known for having harmonic eigenfrequencies are conical and cylindrical. Because of this, these shapes have been widely used for woodwind musical instruments. Other, non continuous, shapes are shown here to have the same property: they consist of a succession of truncated cones (or cylinders) of equal length, which are defined by three initial values for the radii (e.g. the input and output radii of the first cone and the input radius of the second one). The recurrence relations are obtained in the frequency domain, the principle being to impose the existence of travelling waves at the nodes of the lattice: the successive reflections at discontinuities are cancelled at the nodes (but only there). Several kinds of boundary conditions are studied. For the "closed-open" conditions, the unique solution is based upon cylinders and the input impedance curves and its inverse Fourier Transform are shown to have interesting properties. Measurements have been made for this case and the agreement between experiment and theory is satisfactory.

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1. Introduction

It is well known, since Bouasse (1929) and Benade (1976), that a very important requirement for musical wind instruments is the harmonicity of the resonance frequencies: this requirement is demanded by both the definition of musical scales and the necessity of easy intonation (see e.g. Gilbert & Dalmont (1992)). Cylindrical tubes have harmonically related resonance frequencies, when dispersion due to visco-thermal effects or radiation is not too important. Similarly, "open"-open truncated cones (i.e. truncated cones open at one end and excited by a flute-like mechanism) have the same property. On the contrary, "closed"-open truncated cones (excited by reeds), have approximately harmonic resonance frequencies only if the distance from the apex to the input is small compared to the wavelength. For wind instruments, with a more complicated shape and including toneholes, it is an important task for the designer to reduce the anharmonicity of the resonance frequencies.

An interesting theoretical question is therefore posed: is it possible to find shapes of horns, defined as tubes with variable cross-section, including discontinuities in cross section and/or taper, having harmonically related eigenfrequencies (or resonance frequencies), different from a simple cylinder or truncated cone ?

In this context Dalmont (1992), one of the present authors, obtained the following result: a succession of cylindrical tubes of the same length and with an appropriate relation between their cross sections may be considered as a lattice with harmonically related eigenfrequencies (see Figure 1). As a consequence, it is possible to find intermediate shapes of horns between the cylinder and the cone for which the eigenfrequencies are harmonically related.

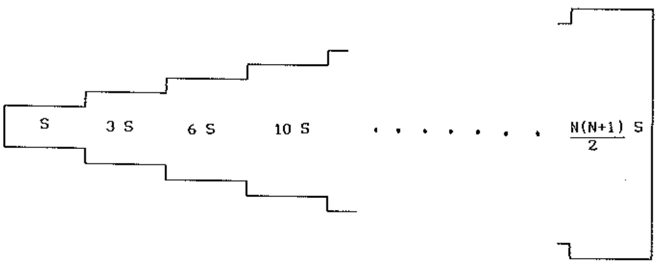


Figure 1. Lattice of cylindrical tubes having harmonically related eigenfrequencies (closed-open case). S is the cross section of the first cylinder and N the number of cylinders.

This result was obtained when the approximation of matched plane waves is valid, the condition of validity being that the transverse dimensions are much smaller than the wavelength. The approximate modelling ignores losses, inductances of step discontinuities and radiation impedances for open tubes, but a more realistic model, as will be shown in the present paper, confirms the result.

The aim of this paper is first to extend the investigation to other possible shapes of lattices of tubes having harmonically related eigenfrequencies, and second to discuss in detail some properties of lattices of cylindrical tubes. In Sections 2 and 3, horns with harmonically related eigenfrequencies corresponding to boundary conditions of zero pressure at the extremities are considered. The discussion of Section 2 is limited to continuous horns, and in Section 3, it is extended to the case of horns with discontinuities (in cross section and its first derivative). We have found a very wide class of horns with the required property, although we have not proved that we have covered all possibilities.

Section 4 deals with the geometrical analysis of the lattices obtained in Section 3, and Section 5 with the calculation of characteristic impedances and transfer matrices. In Section 6 we consider the eigenfrequencies corresponding to other boundary conditions, especially zero volume velocity at the input: for musical wind instruments, zero volume velocity corresponds to reed instruments, whereas zero pressure at the input corresponds approximately to flute like instruments.

Finally, in Section 7, the input impedance and the envelope curves for the peaks and dips are calculated and measured for the simplest case of a lattice built with only cylindrical tubes. In Section 8 interesting time domain properties are discussed.

2. Horns with harmonically related eigenfrequencies for continuous values of the axial coordinate

Consider the classical plane wave horn equation (without losses), due to Lagrange, in the frequency domain (see e.g. Eisner (1967), Benade & Jansson (1974)):

$$(\mathrm{d}p)'' + \left(k^2 - d''/d\right)(\mathrm{d}p) = 0 \tag{1}$$

where $p = p(x)$ is the acoustic pressure, $d = d(x)$ the diameter of the horn, $k = \omega/c$, ω the angular frequency, c the speed of sound in free space. The double prime here indicates the second derivative with respect to the space coordinate, x, corresponding to a wavefront assumed to be planar, and the single prime later indicates the first derivative. In the entire paper, except in Section 8, the time dependence $\mathrm{e}^{j\omega t}$ of the acoustic quantities is omitted. We search for the eigenfrequencies of a horn, with the following, simple boundary conditions:

$p = 0$ for $x = 0$ and $x = X$,
where X is the length of the horn.

Only one profile $d(x)$ allows harmonically related frequencies for every value of X: it is satisfied by $d'' = 0$, i.e. it is a conical horn. So the eigenfrequencies series is the complete series $\omega_i = i\pi c/X$, where i is an integer, and the eigenfunctions are as follows:

$$p_i(x) = \sin i\pi x/X.$$

Obviously, the condition of zero pressure at an end is seldom encountered for gases: nevertheless a radiation impedance, or more precisely a radiation reactance, which has an influence on the resonance frequencies, can be classically taken into account as a "length correction", nearly independent on frequency. As a consequence, even for cones, one could deduce this correction from the total length, and assume a zero impedance at the end. In the case for which experiment have been made, we discuss this question in Section 7.

3. Horns with harmonically related frequencies for discrete values of X

In the previous Section, we have considered a very restrictive condition: the eigenfrequencies are harmonically related for every value of X. We now search for shapes of horns, which may include discontinuities for the functions $d(x)$ and $d'(x)$, for which the eigenfrequencies are harmonically related for particular values of X, x_n, where n is an integer, i.e. for the following boundary conditions:

$p = 0$ for $x = 0$ and $x = x_n$ for a given n

The abscissae values x_n are called the nodes of the lattice. In order to generalize the previous case, the horn is assumed to be continuous and conical between x_{n-1} and x_n (there are probably no other solutions than cones, but this may be disputed). So, using classical results for the transfer matrix of conical horns (see e.g. Benade (1988) or Ayers et al. (1985)), we obtain an expression for the acoustic flow U_n as a function of quantities and dimensions of the horn between x_{n-1} and x_n:

$$jU_n\, 4\rho c/\pi = -\left[d_{n\mathrm{L}}^2 \cot k\ell_\mathrm{L} + \frac{d_{n\mathrm{L}}\left(d_{n\mathrm{L}} - d_{n-1,\mathrm{R}}\right)}{k\ell_\mathrm{L}}\right] p_n + \frac{d_{n-1,\mathrm{R}}\, d_{n\mathrm{L}}}{\sin k\ell_\mathrm{L}}\, p_{n-1} \tag{2L}$$

where U_n is the flow at x_n, p_{n-1} and p_n the pressure at x_{n-1} and x_n, respectively, $\ell_\mathrm{L} = x_n - x_{n-1}$, and $d_{n\mathrm{L}}$ and $d_{n\mathrm{R}}$ the values of the diameter at the left and at the right of x_n, respectively (see Figure 2). The use of the quantities p and U is convenient because they are continuous, even when discontinuities in $d(x)$ or $d'(x)$ occur.

In the same way we can obtain an expression for the truncated cone at the right hand side of x_n:

$$jU_n 4\rho c/\pi = \left[d_{n\mathrm{R}}^2 \cot k\ell_\mathrm{R} - \frac{d_{n\mathrm{R}}\left(d_{n+1,\mathrm{L}} - d_{n\mathrm{R}}\right)}{k\ell_\mathrm{R}}\right] p_n - \frac{d_{n+1,\mathrm{L}}\, d_{n\mathrm{R}}}{\sin k\ell_\mathrm{R}}\, p_{n+1} \tag{2R}$$

where $\ell_\mathrm{R} = x_{n+1} - x_n$.

By eliminating U_n between equations (2L) and (2R), one obtains a relation, between the nodes p_{n-1}, p_n, p_{n+1}.

Now we can search for a solution of equation (1), extended by writing the continuity of pressure and flow at discontinuities, in the following form, valid at x_n:

$$p_n = a_n\left(p^+ \mathrm{e}^{-jkg_n} + p^- \mathrm{e}^{+jkg_n}\right) \tag{3}$$

where a_n and g_n are real quantities, independent of the frequency, and p^+ and p^- are complex coefficients. The eigenfrequencies for the portion (x_n, x_{n+1}) are given by: $\sin k\left(g_{n+1} - g_n\right) = 0$, and are the complete series:

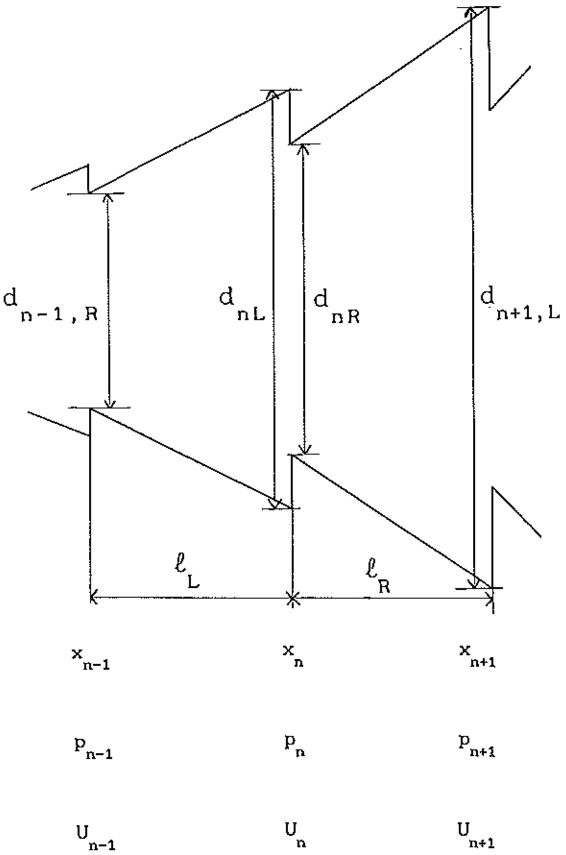


Figure 2. Notations for the lattice of truncated cones.

$\omega_i = i\pi c/(g_{n+1} - g_n)$, where i is an integer. A necessary condition for the existence of a solution as given by equation (3) is that a travelling wave $p_n = a_n\ \mathrm{e}^{-jkg_n}$ is a solution of this equation obtained from equations (2L) and (2R). After dividing by the factor e^{-jkg_n}, this equation can be written in the form:

$$\begin{aligned} a_n \Bigg[& d_{nR}^2 \cot k\ell_R + d_{nL}^2 \cot k\ell_L \\ & - \frac{d_{nR}(d_{n+1,L} - d_{nR})}{k\ell_R} + \frac{d_{nL}(d_{nL} - d_{n-1,R})}{k\ell_R} \Bigg] \\ = & \frac{d_{n+1,L} d_{nR}}{\sin k\ell_R} a_{n+1}\ \mathrm{e}^{-jk[g_{n+1} - g_n]} \\ & + \frac{d_{n-1,R} d_{nL}}{\sin k\ell_L} a_{n-1}\ \mathrm{e}^{-jk[g_{n-1} - g_n]}. \end{aligned} \quad (4)$$

The imaginary part of this equation (corresponding to the energy flow conservation equation) is:

$$\begin{aligned} d_{n+1,L} d_{nR} a_{n+1} \frac{\sin k(g_{n+1} - g_n)}{\sin k\ell_R} = \\ = d_{n-1,R} d_{nL} a_{n-1} \frac{\sin k(g_n - g_{n-1})}{\sin k\ell_L}. \end{aligned}$$

Because it is valid at every frequency , one has:

$$\begin{aligned} g_{n+1} - g_n &= \ell_R = x_{n+1} - x_n \\ g_n - g_{n-1} &= \ell_L = x_n - x_{n-1} \\ d_{n+1} d_{nR} a_{n+1} &= d_{n-1} d_{nL} a_{n-1}. \end{aligned}$$

The energy flow, I, is:

$$I = \frac{1}{2} \mathrm{Re}\ (p_n\ U_n^*) = \frac{1}{2} \frac{d_{n-1,R} d_{nL}}{\sin k\ell_L} \frac{\pi}{4\rho c} \mathrm{Im}\ (p_{n-1} p_n^*)$$

$$\left.\begin{aligned} \text{or} \quad I &= \frac{\pi}{8\rho c} d_{n-1,R} d_{nL} a_{n-1} a_n \\ \text{and} \quad I &= \frac{\pi}{8\rho c} d_{n+1,L} d_{nR} a_n a_{n+1}. \end{aligned}\right\} \quad (5)$$

The * indicates the complex conjugate.

From the real part of equations (4) and (5), one obtains:

$$\frac{a_n^2}{I} \frac{\pi}{8\rho c} = \frac{\cot k\ell_R + \cot k\ell_L}{d_{nR}^2 \cot k\ell_R + d_{nL}^2 \cot k\ell_L - \frac{d_{nR}(d_{n+1,L} - d_{nR})}{k\ell_R} + \frac{d_{nL}(d_{nL} - d_{n-1,R})}{k\ell_L}} \quad (6)$$

This equation is valid for every value of the frequency. So one deduces a first geometrical condition:

$$d_{nR}(d_{n+1,L} - d_{nR})/\ell_R = d_{nL}(d_{nL} - d_{n-1,R})/\ell_L \quad (7a)$$

An elementary calculation leads to the following, equivalent result:

$$S'_{nR} = S'_{nL} \quad (7b)$$

where $S = \pi d^2/4$ is the cross section, and S' its derivative with respect to x.

The other condition obtained from equation (6) is the independence with respect to frequency of the following quantity:

$$K = (\cot k\ell_R + \cot k\ell_L) / (d_{nR}^2 \cot k\ell_R + d_{nL}^2 \cot k\ell_L)\cdot$$

Two cases should be considered:

i) if $d_{nR} = d_{nL}$ (or $S_{nR} = S_{nL}$), the condition is valid for any length: it is the case of a continuous conical horn, as treated in Section 2.

ii) if $d_{nR} \neq d_{nL}$, the lengths ℓ_R and ℓ_L are equal because of the independence of K with respect to frequency, and it follows from equations (6) and (7a):

$$\frac{a_n^2 \pi}{I 8\rho c} = \frac{1}{d_{nR}^2 + d_{nL}^2} \quad (8a)$$

or

$$a_n^2 = 4I\pi c/(S_{nR} + S_{nL})\cdot \quad (8b)$$

If this result is used for a_{n-1}, one obtains, after some algebra, from equations (5) and (8):

$$4\ d_{n-1,R}^2 d_{nL}^2 = (d_{nL}^2 + d_{nR}^2)(d_{n-1,L}^2 + d_{n-1,R}^2) \quad (9a)$$

or

$$(S_{nL} + S_{nR})(S_{n-1,R} + S_{n-1,L}) = 4 S_{nL} S_{n-1,R}\cdot \quad (9b)$$

For a semi-infinite lattice ($n \geq 0$), equations (7) and (9) define recurrence relationships, the lattice being defined by three initial dimensions: d_{0R}, d_{1L}, d_{1R}. These recurrence relationships can be transformed as explained in Section 4.

These developments correspond to a lattice made with truncated cones, as it has been shown, of equal length and with appropriate step discontinuities between them.

Another possible line of development would be to use the equivalent electrical circuits of conical tubes (see e.g. Benade (1988)). An equivalent circuit for a conical tube is the classical circuit for a cylindrical tube with shunt inductances at both ends. The basic idea of our treatment is to compensate the effect of the shunt inductance at the input of a conical tube with the effect of the shunt inductance at the output of the previous tube.

4. Geometry of the lattice

From equations (7) and (9), one obtains the following, general formulae:

$$\begin{cases} d_{n\mathrm{R}}^2 = d_n^2\left[1+\beta/(1+n\beta)\right] \\ d_{n\mathrm{L}}^2 = d_n^2\left[1-\beta/(1+n\beta)\right] \end{cases} \tag{10}$$

where

$$d_n^2 = \frac{2d_{0\mathrm{R}}^2 d_{1\mathrm{L}}^2}{d_{1\mathrm{L}}^2 + d_{1\mathrm{R}}^2}(1+n\alpha)^2 = \left(d_{n\mathrm{L}}^2 + d_{n\mathrm{R}}^2\right)/2 \tag{11}$$

and

$$\alpha = \left(d_{1\mathrm{L}}^2 + d_{1\mathrm{R}}^2\right)/2d_{0\mathrm{R}}d_{1\mathrm{L}} - 1;$$
$$\beta = \left(d_{1\mathrm{R}}^2 - d_{1\mathrm{L}}^2\right)/2d_{1\mathrm{L}}^2.$$

The analysis of equations (10) and (11) can be made by considering successively 3 parameters: α, β, and $\alpha-\beta$, which are related to the "generating cone", the rate of step discontinuities, and the conicity of a cell, respectively, as follows (see Figure 3):

i) Equation (11) defines a "generating" cone, with a conicity proportional to α. If α is positive, the cone is divergent. If α is negative, the generating cone is convergent. So in that case it is finite because the quantity $1+n\alpha$ must remain positive in equation (11) (we notice that a cone convergent then divergent - i.e. $(1+n\alpha)$ at first positive then negative - can be solution of equation (10), but it can be shown that these equations are not valid in this case). Finally, if α is infinite, the generating cone has its apex at $n=0$ ($d_\mathrm{R}=0$) and equation (11) becomes $d_n^2 = d_1^2 n^2$. In the general case, when $n \to \infty$, the lattice tends to the generating cone.

ii) The rate of step discontinuity, $d_{n\mathrm{R}}^2/d_{n\mathrm{L}}^2$, is given by the equation obtained from equation (10):

$$\frac{d_{n\mathrm{R}}^2}{d_{n\mathrm{L}}^2} = \frac{1+(n+1)\beta}{1+(n-1)\beta}$$

and depends on one parameter only, β. The sign of $(d_{n\mathrm{R}}^2/d_{n\mathrm{L}}^2 \ -1)$ is independent of n. For $\beta=0$ the rate is equal to unity and the lattice is a cone without discontinuities, i.e. it is equal to the generating cone (Figure 3b).

If β is negative ($d_{1\mathrm{R}} < d_{1\mathrm{L}}$), the ratio $d_{n\mathrm{R}}/d_{n\mathrm{L}}$ is less than unity and increases with n. The conicity of a cell is larger than the conicity of the generating cone (Figure 3a). On the contrary, if β is positive ($d_{1\mathrm{R}} > d_{1\mathrm{L}}$), the ratio $d_{n\mathrm{R}}/d_{n\mathrm{L}}$ is larger than unity and decreases with n. The conicity, then, is less than the conicity of the generating cone (Figures 3c, 3d and 3e).

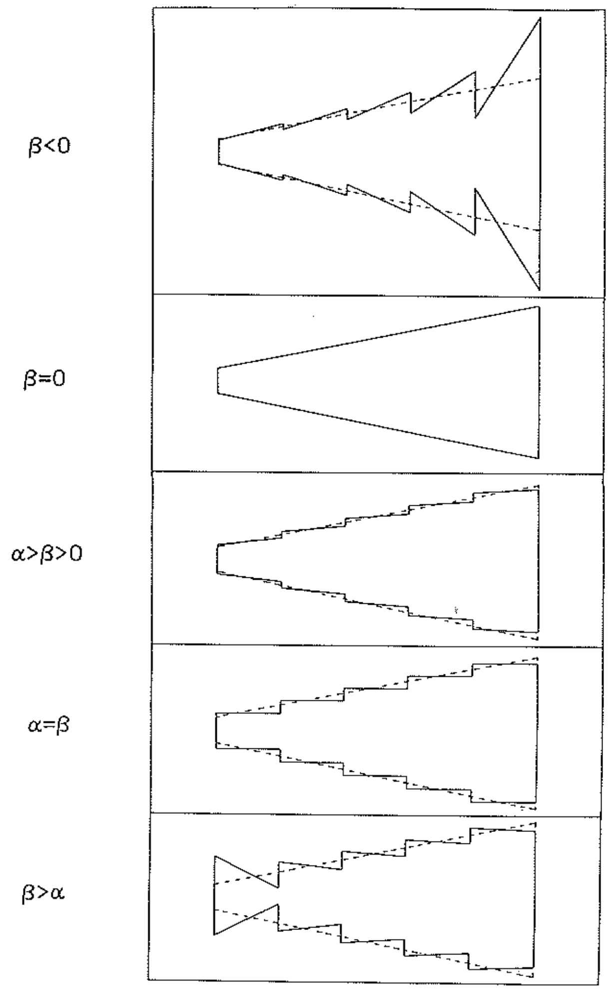


Figure 3. The different cases of lattices having harmonically related eigenfrequencies for positive α and increasing values of β. For each case $N=5$ and $\alpha=1$. β is equal to -0.1837, 0., 0.3333, 1. and 9., respectively for Figures a), b), c), d), e). Dotted lines: generating cone.

If β is negative, the lattice is finite, because finite and positive values of $d_{n\mathrm{L}}^2$ imply $n < -1/\beta$. Because $2\beta+1$ must be positive, the maximum value N of n is greater or equal to 2.

iii) The conicity of a cell, equal to $\left(d_{n+1,\mathrm{L}} - d_{n\mathrm{R}}\right)/\ell$, is proportional to the quantity $(\alpha-\beta)$, and its sign is independent of n and equal to the sign of the conicity of the first cell. For $\alpha-\beta=0$ ($d_{0\mathrm{R}} = d_{1\mathrm{L}}$), all cells are cylindrical (Figure 3d).

5. Characteristic impedances and transfer matrices of the lattice

We consider a travelling wave: $p_n = a_n \mathrm{e}^{\mp jkn\ell}$. This wave travels like a lattice wave, i.e. a wave existing only for discrete abscissae. But, between two abscissae x_n and x_{n+1}, it is easy to show that the step discontinuities produce reflections: if $p_n = a_n \mathrm{e}^{-jkn\ell}$ and

$p_{n+1} = a_{n+1}e^{-jk(n+1)\ell}$, $p(x)$ is a superposition of two waves, e^{-jkx} and e^{+jkx}. We discuss this fact in Section 8.

From equations (2L) (or 2R) and (8), we obtain the characteristic admittance Y_n of this travelling wave ($Y_n^\pm = U_n/p_n$):

$$\begin{aligned} Y_n^\pm \rho c = \; & -j\frac{1}{2}\left(S_{nR} - S_{nL}\right)\ \cot k\ell \\ & +j\frac{1}{2}\frac{S_n'}{k} \pm \frac{1}{2}\left(S_{nR} + S_{nL}\right) \end{aligned} \tag{12}$$

for

$$\frac{p_{n+1}}{p_n} = \frac{a_{n+1}}{a_n}e^{\mp jk\ell}.$$

Without discontinuities ($S_{nR} = S_{nL}$), this formula is the classical formula for the characteristic admittance of a spherical wave in a cone. Thus, the general solution can be written (from equation (3)):

$$\begin{aligned} p_n &= a_n\left(p^+e^{-jkn\ell} + p^-e^{jkn\ell}\right) \\ U &= a_n\left(p^+Y_n^+e^{-jkn\ell} + p^-Y_n^-e^{jkn\ell}\right) \end{aligned}$$

Eliminating p^+ and p^-, we can deduce that a formula for the transfer matrix between two different nodes x_n and x_m can be written as follows:

$$\begin{aligned} \begin{pmatrix} p_n \\ U_n \end{pmatrix} &= \begin{pmatrix} z_{cn} & 0 \\ 0 & z_{cn}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y_nZ_{cn} & 1 \end{pmatrix} \\ &\times \begin{pmatrix} \cos k\left(x_m - x_n\right) & j\ \sin k\left(x_m - x_n\right) \\ j\ \sin k\left(x_m - x_n\right) & \cos k\left(x_m - x_n\right) \end{pmatrix} \\ &\times \begin{pmatrix} 1 & 0 \\ -y_mZ_{cm} & 1 \end{pmatrix} \begin{pmatrix} z_{cm}^{-1} & 0 \\ 0 & z_{cm} \end{pmatrix} \begin{pmatrix} p_m \\ U_m \end{pmatrix} \end{aligned} \tag{13}$$

where $z_{cn} = Z_{cn}^{1/2}$, $Z_{cn} = \rho c/S_n$, and

$$y_nZ_{cn} = -j\frac{S_{nR} - S_{nL}}{S_{nR} + S_{nL}}\ \cot k\ell + \frac{1}{2}j\frac{S_n'}{kS_n}$$

and similar definitions hold for the subscript m.

6. Lattices with harmonically related eigenfrequencies for various boundary conditions

In the previous sections lattices with harmonically related eigenfrequencies were found with the boundary conditions of zero pressure at both ends of the lattice. By analogy with the closed-open cylinder (reed instruments) for which the eigenfrequencies are in the ratios 1, 3, 5,..., we now try to study the problem of mixed boundary conditions: $p = 0$ for $m = N$, and a different condition for $n = 0$.

If $p = 0$ for $m = N$, from equation (13), one obtains the input admittance:

$$\begin{aligned} \frac{j\rho cY_0}{S_{0R}} &= \left(1 - \frac{d_{1L}^2}{d_1^2}\right)\ \cot k\ell \\ &+ \frac{1}{k\ell}\left(\frac{d_{1L}}{d_{0R}} - 1\right) + \frac{d_{1L}^2}{d_1^2}\ \cot kL \end{aligned} \tag{14}$$

where $L = N\ell$.

If $p = 0$ for $n = 0$, the admittance $Y_0 = \infty$, and we find as expected $\cot kL = \infty$. We notice that $\cot k\ell = \infty$ is a second solution, but it is included in the first solution. If the first cone is complete, i.e. $d_{0R} = 0$, we obtain the same solution in order to have a finite pressure (Y_0/S_{0R} is the ratio pressure/velocity). This case is similar to that of the complete cone (see for a discussion Ayers et al. (1985)).

We are now interested in lattices closed at $x = 0$ ($Y_0 = 0$), with $d_{0R} \neq 0$. It is well known that for conical reed woodwind instruments (see e.g. Nederveen (1969)) consisting approximately of truncated cones, the resonance frequencies are close to the frequencies of a complete cone. This approximation is valid when the distance from the apex to the input of the truncated cone is small compared to the wavelength. We can then search for a generalization, using equation (14).

If $k\ell \ll 1$, equation (14) can be rewritten as follows:

$$\frac{j\rho cY_0}{S_{0R}} = \frac{1}{k\ell}\left(\frac{d_{1L}}{d_{0R}} - \frac{d_{1L}^2}{d_1^2}\right) + \frac{d_{1L}^2}{d_1^2}\ \cot kL.$$

So the eigenfrequencies for $Y_0 = 0$ are given by:

$$\cot kL + \alpha/k\ell = 0.$$

If ℓ/α is small compared to the wavelength, one obtains:

$$f_n = \frac{nc}{2(L + \ell/\alpha)}.$$

This formula is valid if both ℓ and ℓ/α are small compared to the wavelength. The first condition is satisfied for the first eigenfrequencies if N is sufficiently large. The second condition implies that the distance from the apex of the generating cone to its input is sufficiently small compared to the wavelength.

Equation (14) suggests other cases with harmonically related eigenfrequencies for zero admittance and without the low frequency condition. If the lattice is built with cylindrical cells ($d_{1L} = d_{0R}$), and if the coefficients of the terms $\cot k\ell$ and $\cot kL$ are equal ($d_1^2 = 2d_{1L}^2$), then equation (14) becomes:

$$\cot k\ell + \cot kL = 0$$

or

$$\sin k(\ell + L)/(\sin k\ell\ \sin kL) = 0 \tag{15}$$

The eigenfrequencies are given by: $f_i = ic/2L'$, where $L' = L + \ell$, except if i is a multiple of $L'/\ell = 1 + N$. In this case, $\alpha = \beta = 1$, and equations (10) and (11) become:

$$\begin{aligned} d_n^2 &= d_0^2(n+1)^2, \\ d_{nR}^2 &= d_{0R}^2(n+1)(n+2)/2, \\ d_{nL}^2 &= d_{0R}^2 n(n+1)/2. \end{aligned} \tag{16}$$

Notice that $d_0^2 = d_{0R}^2/2$, and $d_{nR} = d_{n+1,L}$. In this case, $d_{0L} = 0$, i.e. if the lattice is extended on the left, the first value is zero: the boundary condition is in accordance with the formulae defining the lattice. Another interpretation is the following: the lattice has a length L', and includes a zero diameter cylinder ($d_{-1,R} = d_{0L} = 0$).

For every design, the series of harmonic frequencies is incomplete. For $N = 1$ ($L = \ell$), the lattice is a single cylinder, and the eigenfrequencies ratio series is 1, 3, 5, 7... For $N = 2$, the series is 1, 2, 4, 5, 7, 8, 10, 11 ... For $N = 3$, the series is 1, 2, 3, 5, 6, 7, 9, 10, 11, ... If N tends to infinity, the series tends to become complete, and the lattice tends to be to a complete cone.

As a conclusion, we have found the intermediate cases between a closed cylinder and a complete cone, as explained by Dalmont (1992).

7. Input impedance of a finite lattice of cylindrical tubes

It is interesting to study the input impedance curves of lattices of cylindrical tubes satisfying equation (16) (with zero end impedance), i.e. to consider their response to forced oscillations instead of their free oscillation behaviour. In these calculation, it is necessary to take into account losses due to visco-thermal effects and radiation. It is possible to use a rather precise model as did Kergomard (1981) for the calculation of resonances in horns, but we prefer to simplify the model: we assume that the visco-thermal losses are independent of the radius, the wavenumber k being replaced by $k_{\rm vt} = k - j\alpha_{\rm vt}$ (we neglect the visco-thermal effects in the characteristic impedances), and we use the low frequency value for the radiation impedance $Z_{\rm R}$ (see Levine & Schwinger (1948)):

$$Z_{\rm R} = \frac{\rho c}{S}\frac{1}{4}(kr)^2$$

For this calculation, our model ignores the step discontinuities (see e.g. Kergomard & Garcia (1987) and Kergomard (1991)) and, for coherence, the radiation inductance (these quantities will be taken into account later for comparison with experiments, their influence being discussed further). We use equation (13), with $n = 0$ and $m = N$, to calculate the input impedance Z_0 as a function of the radiation impedance, and obtain, after some algebra:

$$\frac{Z_0S_0}{\rho c} = j \sin k_{\rm vt}\ell$$

$$\times \frac{\frac{Z_{\rm N}S_0}{\rho c}(N+1)[(N+1)\cos k_{\rm vt}L - \sin k_{\rm vt}L \cot k_{\rm vt}\ell] + j\sin k_{\rm vt}L}{\frac{Z_{\rm N}S_0}{\rho c}(N+1)[(N+1)\cos k_{\rm vt}L' - \sin k_{\rm vt}L' \cot k_{\rm vt}\ell] + j\sin k_{\rm vt}L'} \quad (17)$$

where

$$L = N\ell,\ L' = (N+1)\ell,\ S_0 = \pi d_0^2/4 = \pi d_{0\rm R}^2/8,$$

and

$$Z_N = \frac{\rho c}{S_{NL}}\frac{1}{4}(kr_{NL})^2$$

or

$$\frac{Z_NS_0}{\rho c} = \frac{1}{N(N+1)}\frac{1}{4}(kr)^2$$

with

$$r = r_{NL}.$$

The calculation of the extrema of Z_0 is made at the first order of $(kr)^2$ and $\alpha_{\rm vt}\ell$. The values of the resonance and antiresonance frequencies are obtained from the calculation when losses are neglected (see equation (15)), viz.:

$$\frac{Z_0S_0}{\rho c} = j\frac{\sin k\ell \sin kL}{\sin kL'}. \quad (18)$$

Three cases need to be considered, one for the resonances and two for the antiresonances:

i) **Resonances**: $\sin kL' = 0$, $\sin k\ell \neq 0$.

The quantities $\sin k_{\rm vt}\ell$ and $\sin k_{\rm vt}L$ are of the zeroth order of the losses, and $\sin k_{\rm vt}L'$ of the first order:

$$\sin (k_{\rm vt}L') = \sin ((k - j\alpha_{\rm vt})L') = -j\alpha_{\rm vt}L' \cos kL'.$$

The result is:

$$\frac{Z_0S_0}{\rho c} = j \sin k\ell \times \frac{j \sin kL}{\cos kL'\left[\frac{Z_NS_0}{\rho c}(N+1)^2 + \alpha_{\rm vt}L'\right]}$$

or, because $\sin kL = \sin (k(L' - \ell)) = -\cos kL' \sin k\ell$,

$$Z_0 = \frac{\rho c}{S_0}\frac{\sin^2 k\ell}{\frac{N+1}{N}\frac{1}{4}(kr)^2 + \alpha_{\rm vt}(N+1)\ell}. \quad (19i)$$

The numerator $\sin^2 k\ell$ is equal to:

- $\sin^2 i\pi/2 = 1$ if $N = 1$ (single cylinder, $i \neq 2n$)
- $\sin^2 i\pi/3 = 3/4$ if $N = 2$ ($i \neq 3n$)
- $\sin^2 i\pi/4 = 1/2$ or 1 if $N = 3$ ($i \neq 4n$)

(i and n being integers).

For m infinite and for a given total length L, the result is consistent with that for a truncated cone where the length ℓ tends to zero, so $\sin^2 k\ell \simeq k\ell^2$. For a truncated cone (see Kergomard (1981)), the equivalent factor is:

$$\left(1 + \frac{1}{k^2x_0^2}\right)^{-1} \simeq k^2x_0^2,$$

where x_0 is the distance from the apex to the input of the truncated cone.

ii) **Antiresonance of the first kind**: $\sin kL = 0$, $\sin k\ell \neq 0$

The quantities $\sin k_{\rm vt}\ell$ and $\sin k_{\rm vt}L'$ are of the zeroth order, and $\sin k_{\rm vt}L$ of the first order:

$$\sin k_{\rm vt}L = -j\alpha_{\rm vt}L \cos kL.$$

The result is:

$$\frac{Z_0S_0}{\rho c} = j \sin k\ell \frac{\frac{Z_NS_0}{\rho c}(N+1)^2 + \alpha_{\rm vt}L \cos kL}{j \sin kL'}$$

or

$$Z_0 = \frac{\rho c}{S_0}\left[\frac{N+1}{N}\frac{1}{4}(kr)^2 + \alpha_{\rm vt}N\ell\right]. \quad (19ii)$$

This result is consistent with that for a truncated cone: there is no "reactive" factor, and the envelope curve for the minima has a shape similar to cylinders.

iii) **Antiresonances of the second kind**: $\sin k\ell = 0$

All quantities $\sin k_{vt}\ell$, $\sin k_{vt}L$, and $\sin k_{vt}L'$ are of the first order:

$$\frac{Z_0 S_0}{\rho c} = \alpha_{vt}\ell \frac{\cos k\ell \cos kL}{\cos kL'} \times \frac{\frac{Z_N S_0}{\rho c}(N+1)[N+1-L/\ell] + \alpha_{vt}L}{\frac{Z_N S_0}{\rho c}(N+1)[N+1-(L+\ell)/\ell] + \alpha_{vt}(L+\ell)}$$

or

$$Z_0 = \frac{\rho c}{S_0}\left[\frac{1}{4}(kr)^2 + \frac{N}{N+1}\alpha_{vt}\ell\right]. \qquad (19iii)$$

This case does not correspond to any case of a truncated cone. We notice that for these antiresonances, the input impedance is very small compared with the impedances of the antiresonances of the first case, and is very small compared with the antiresonances of a cylinder of the same length.

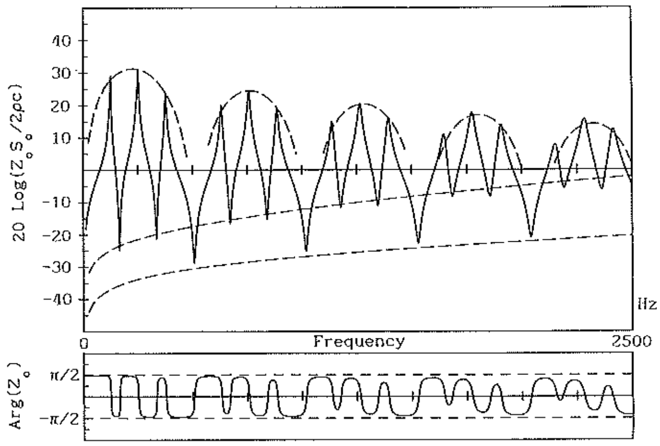


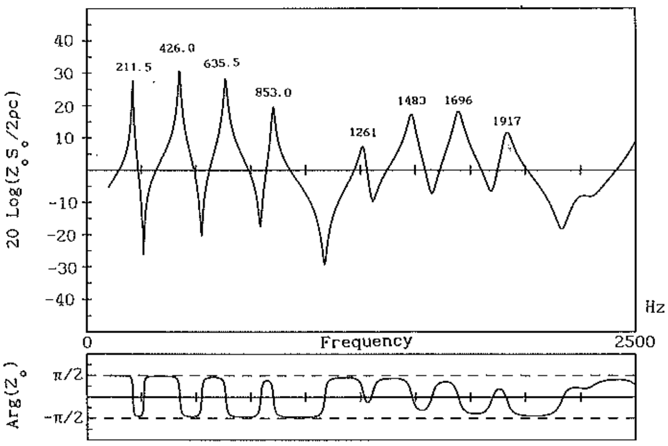
Figure 4. Calculated input impedance of a lattice of 3 cylinders with a total length 1 meter and radii 8 mm, 13.86 mm and 19.6 mm, respectively (full lines). Dotted lines give the envelope curves calculated with the simplified theory (equations (19)).

Figure 4 shows the input impedance curve calculated with a "complete" model (i.e. a model taking into account losses - see Keefe (1984) or Bruneau et al. (1989) -, step discontinuity inductances - see Kergomard & Garcia (1987) and Kergomard (1991) - and radiation impedance -see Levine & Schwinger (1948) -) for a lattice of 3 cylinders with the following parameters: diameters 8 mm, 13.86 mm, 19.8 mm, total length $L = 1$ m. The approximate formulae (19) predict the heights of the extrema with a satisfying accuracy at low frequencies.

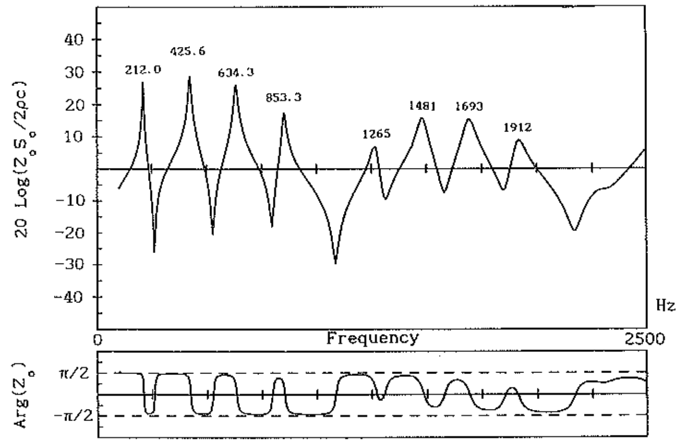
It is interesting to examine how the model calculated without dissipation and without discontinuities and radiation inductances remains valid when these phenomena are taken into account.

If, as an example, we calculate, with a "complete" model, a lattice made with four cylinders of length 155 mm and of input diameter 14 mm, the four first resonance frequencies are equal to 211.7 Hz, 429.8 Hz, 645.8 Hz and 863.3 Hz respectively at 20 °C. This leads to an anharmonicity of 0.1%, 0.3% and 0.5% between the first resonance frequency and respectively the second, third and fourth resonance frequency. Anharmonicity is more important at higher frequencies: 0.3%, 0.5%, 0.6% and 0.8% between the first resonance frequency and respectively resonances number 5, 6, 7, 8.

It can also be shown that if the last tube is longer or shorter than the others, or if one added a short tube to the last one, the anharmonicity of the first $(N-1)$ peaks (N number of cylinders) remains very small. As an example, if the previous lattice is extended by a short tube of length 20 mm and diameter 60 mm, the anharmonicity between the first eight peaks remains less than 1%.



a)



b)

Figure 5. Input impedance of a lattice of four cylinders with length 0.155 m and radii 7 mm, 12.4 mm, 17.4 mm and 22.4 mm, respectively, ended in a cylinder of length 18 mm and radius 29 mm. a) Calculation. b) Measurement.

A comparison with experiment has been made for this last horn made with 4 cylinders of length 155 mm and of diameter 14 mm, 24.8 mm, 34.7 mm and 44.7 mm, respectively, terminated in a short cylinder of 18 mm length and 58 mm diameter. A sensor using an electrostatic transducer as a source (Dalmont & Bruneau, 1992) and calibrated with the method described by Dalmont & Herzog (1993) has been used. "Complete" theory (including discontinuities inductance) and experiment are in good agreement, as shown in Figure 5. Accuracies of both measured and theoretical resonance frequency are estimated to about 0.2%. The anharmonicity, even with the added cylinder of length 18 mm, is weak (less than 1%) and well predicted by the theory. Notice that experimental amplitudes are about 2 dB weaker than predicted by the theory. This can be explained by the difficulties in modelling losses, especially at discontinuities.

8. Time domain responses of a lattice of cylindrical tubes

8.1. *Travelling pressure waves in an infinite lattice*

The property of the lattices obtained in Section 3 is very particular: the solution is in the form of travelling waves $p_n = a_n \mathrm{e}^{-jkx_n}$ with free space sound speed. Nevertheless, for such a "travelling" wave, between two nodes of the lattice, the solution is not a single travelling wave, but the superposition of ordinary plane waves travelling in the two opposite directions. (For simplicity, we restrict the following considerations to lattices of cylindrical tubes). For $x_n < x < x_{n+1}$, $p(x)$ is the superposition of two ordinary travelling waves, e^{-jkx} and e^{jkx} thus:

$$\begin{aligned} p(x) = & \frac{\mathrm{e}^{-jk(x-x_n)}}{2j \sin k\ell}\left[p_n \mathrm{e}^{jk\ell} - p_{n+1}\right] \\ & -\frac{\mathrm{e}^{jk(x-x_{n+1})}}{2j \sin k\ell}\left[p_n - p_{n+1}\mathrm{e}^{jk\ell}\right]. \end{aligned} \tag{20}$$

or

$$p(x) = \left[p_{n+1} \sin k\,(x - x_n) - p_n \sin k\,(x - x_{n+1})\right]/\sin k\ell$$

It appears that the single travelling wave form exists only for $x = x_n$ or $x = x_{n+1}$. This fact suggests an interpretation, based on the time domain response, for the conditions defining the lattices found in Section 3. Let us consider an infinite lattice, and the pressure impulse response in x_{n+1} for a pulse at $x = x_n$: $p_n = a_n\delta(t)$. A physical interpretation of this excitation is as follows: after a pulse at $t = 0$, the pressure is zero at $x = x_n$ for $t > 0$. So all waves are reflected with a change of sign at $x = x_n$. The question is why the response at $x = x_{n+1}$ is $p_{n+1} = a_{n+1}\delta(t - \ell/c)$, i.e. a single pulse.

At all step discontinuities, an incoming pulse is divided into two parts: a reflected one, the reflection coefficient being $R^- = (S_\mathrm{L} - S_\mathrm{R})/(S_\mathrm{L} + S_\mathrm{R})$, and a transmitted one, the transmission coefficient being $T^+ = 1 + R^-$. Similarly, an outgoing pulse is divided into two parts, the coefficients being $R^+ = -R^-$ and $T^- = 1 + R^+$. At time $t = 3\ell/c$ and position $x = x_{n+1}$, the amplitude of the pulse is (after elementary considerations on the travel of the initial pulse):

$$R^-_{n+1} \cdot (-1) \cdot T^+_{n+1} + T^+_{n+1} \cdot R^-_{n+2} \cdot T^-_{n+1}.$$

A condition for having zero pressure at $t = 3\ell/c$ and $x = x_{n+1}$ is therefore:

$$R^-_{n+1} = R^-_{n+2} T^-_{n+1} \tag{21}$$

or

$$\begin{aligned} &(S_{n+1,\mathrm{L}}/S_{n+1,\mathrm{R}} - 1)\,(S_{n+2,\mathrm{L}}/S_{n+2,\mathrm{R}} + 1) \\ &= 2\,(S_{n+2,\mathrm{L}}/S_{n+2,\mathrm{R}} - 1)\,. \end{aligned}$$

This equation is identical to equation (9b), and is satisfied in particular for cylindrical tubes defined by equation (16). As a consequence, using a recursive reasoning, we deduce that for x_{n+m}, $p_{n+m}(m\ell/c + 2\ell/c)$ is zero. Then, by a simple (but rather extended) reasoning, it can be deduced that the pressure p_{n+m} is zero at any time after $t = (m+2)\ell/c$. So the response is a single pulse.

As a consequence, we have explained that equation (21) is the basic condition for the production of a travelling wave in the lattice, the successive reflected pulses vanishing only at the nodes of the lattice.

8.2. *Green function of an infinite lattice*

Another interesting property of an infinite lattice of cylindrical tubes can be deduced from the value of the input impedance, obtained from equation (12), using equation (16) and setting $S_0 = \pi d_0^2/4$:

$$Y_0 \rho c = S_0(1 - j \cot k\ell),$$

or

$$Z_0 = \frac{\rho c}{S_0} j \sin k\ell\, \mathrm{e}^{-jk\ell}. \tag{22}$$

Two remarkable consequences of this result are as follows:

i) the inverse FT of the input impedance, $h(t)$, proportional to the time derivative of the Green function at $x = 0$, is finite:

$$h(t) = \frac{\rho c}{2S_0}[\delta(t) - \delta(t - 2\ell/c)].$$

This function is the pressure response to a volume velocity pulse. Using the previous result concerning the travelling wave, one deduces that the pressure response at any node x_n to a volume velocity pulse at $x = 0$ is finite too.

ii) Using both equations (20) and (22), we see that the factor $j \sin(k\ell)$ in the denominator of equation (20) is compensated for the same factor in the numerator of equation (22) in the calculation of the transfer impedance p/U_0 at any position x:

$$\begin{aligned} \frac{p}{U_0} = \frac{\rho c}{2S_0} & \left[\frac{1}{n+1}\left(\mathrm{e}^{-jkx} - \mathrm{e}^{jk[x-2(n+1)\ell]}\right)\right. \\ & \left. -\frac{1}{n+2}\left(\mathrm{e}^{-jkx}\mathrm{e}^{-2jk\ell} - \mathrm{e}^{jk(x-2(n+1)\ell)}\right)\right] \end{aligned} \tag{23}$$

for $x_n < x < x_n + \ell$.

So the pressure response at any position x to a volume velocity pulse $U_0 = \delta(t)$ at $x = 0$ is finite, even between

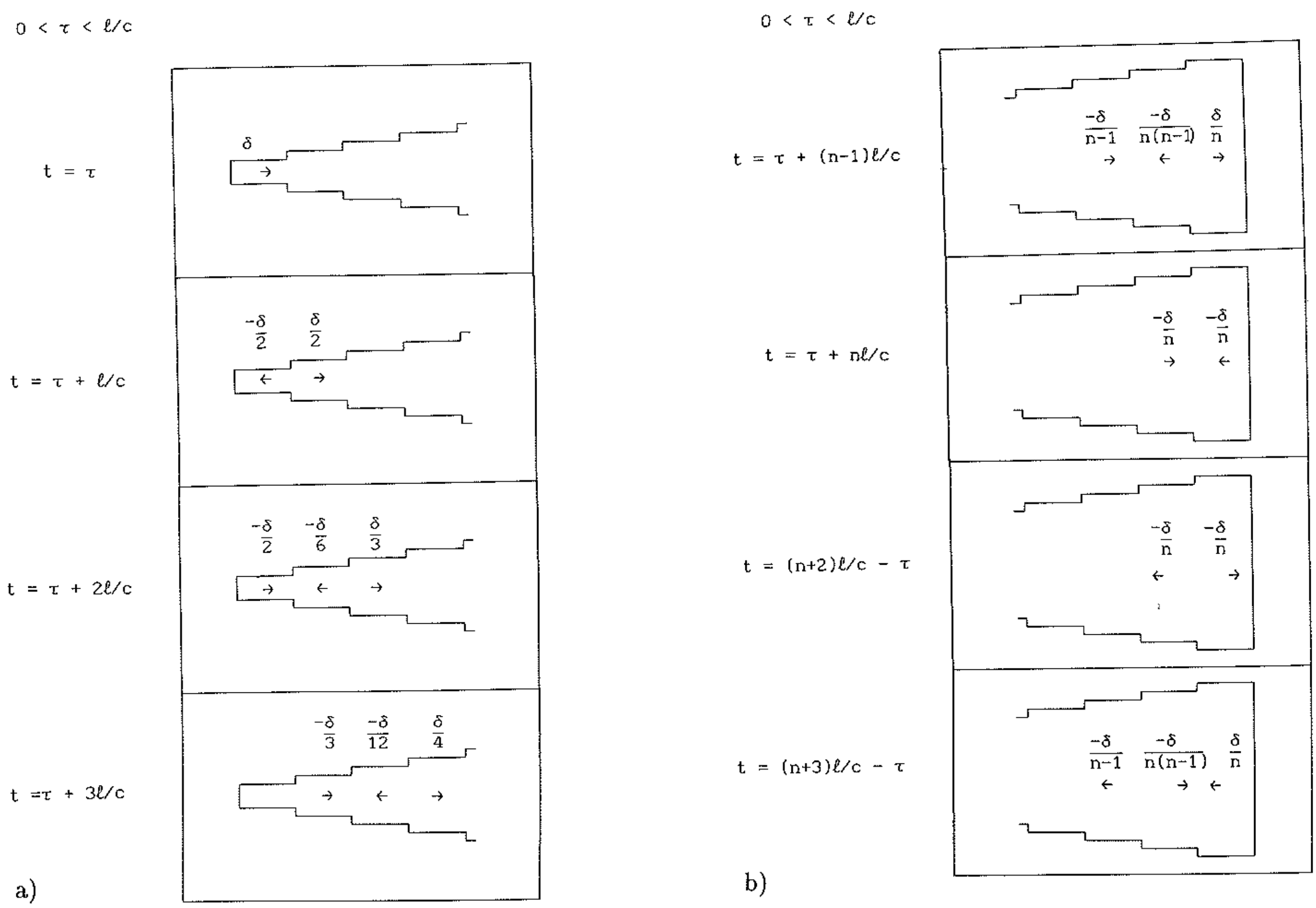


Figure 6. Schematic representation of the propagation on an impulse δ in a lattice of cylinders with harmonically related eigenfrequencies (see Figure 1). The arrows indicate the direction of propagation and the location of the pulses at the considered time. The values above the arrows give the amplitudes of each pulse. a) Evolution from the initial time. b) Reflection at the end.

two nodes. More precisely from rewriting equation (23), this response appears to involve three different pulses:

$$\frac{p}{U_0} = \frac{\rho c}{2S_0} e^{-jkx}\left[\frac{1}{n+1} - \frac{1}{(n+1)(n+2)} e^{2jk[x-(n+1)\ell]} - \frac{1}{n+2} e^{-2jk\ell}\right]$$

or

$$\mathrm{FT}^{-1}\left[\frac{p}{U_0}\right] = \frac{\rho c}{2S_0}\left[\frac{1}{n+1}\delta(t-x/c) - \frac{1}{(n+1)(n+2)}\delta(t-[2(n+1)\ell - x]/c) - \frac{1}{n+2}\delta[t-(x+2\ell)/c]\right]. \quad (24)$$

At the nodes of the lattice, the response is reduced to two pulses. This result is illustrated by the schemes of Figure 6a: the evolution from initial time is shown at every time as a function of space. The relation with equation (24) is elementary, by fixing t in equation (24), and searching for the values of x for which the delta functions are not zero (it is of course necessary to replace n by $E(x/\ell)$, where E is the function integer part). The arrows indicate the directions of propagation of the pulses.

iii) When the lattice is finite (and open at the end, with the condition $p = 0$), the pulses are simply reflected at the end with a change in sign, but the addition of the different pulses leads to the remarkable following result: for a given time, the three incident pulses seem to have been reflected without change in sign, i.e. the location and the amplitude of the three pulses for a time $t = \tau$ is the same as for a time $t' = 2(n+1)\ell/c - \tau$. Only the direction of the propagation differs. This result is illustrated in Figure 6b. In a similar way, the reflection at the input (closed because the source is a pulse of velocity) is consequently identical to the departure after the initial time (Figure 6a).

9. Discussion and conclusion

We have found a general family of horns having harmonically related eigenfrequencies. For the open-open case, the horns are made of a succession of truncated cones of equal length and can be defined from the choice of three initial values for the diameters, d_{0R}, d_{1L}, d_{1R}, i.e.

the dimensions of the first cone and the input diameter of the second one. In this case, the series of the eigenfrequencies is complete. For the closed-open case, we find a generalization of the classical results concerning truncated cones and cylinders. For horns built with truncated cones with sufficiently steep taper, the series is approximately the same as the series for the open-open case. For horns built with N cylinders, however, the series is incomplete: $\omega_i = i\pi/L$ where i is any integer except a multiple of $N+1$.

In this last case, the input impedance and its inverse Fourier Transform have original properties: the peaks envelope of the input impedance is the sine function $\sin^2(k\ell)$, and the FT^{-1} of the transfer impedance $p(x)/U_0$ of an infinite lattice is finite at every position of x in the lattice, even between nodes.

Our first treatment of the problem is in the frequency domain, because of the research of eigenfrequencies. In the time domain, we deduced from the form of the solutions a second possible method of derivation of the result, based on the existence of travelling lattice waves. We limited the time domain discussion to lattices of cylindrical tubes, but an extension to lattices of conical tubes could be possible, using reflection functions at step and/or taper discontinuities between cones (see Agulló et al. (1988) and (1992), Martinez & Agulló (1988) and Gilbert et al. (1990)). Probably we could obtain the same explanation as for cylindrical tubes, i.e. the destructive superposition of successive reflected waves at the nodes of the lattice.

Two kinds of generalization of our results could be studied. On the one hand, the closed-closed case can be studied by simply using the property of duality, as explained by Pyle (1975), pressure and volume velocity being inverted, and the cross section function $S(x)$ being replaced by $1/S(x)$. So the truncated cones should be replaced by parabolic horns.

On the other hand, it is possible to use branched tubes of length ℓ at the nodes of lattices found in Section 3: if they are open (i.e. terminated by a zero impedance), the coefficient $(d_{\mathrm{nR}}^2 + d_{\mathrm{nL}}^2)$ of the quantity cot $(k\ell)$ appearing in equation (4) is modified. Thus a wider class of lattices can be found, but the study presented in this paper is concerned only with lattices built with conical tubes but without branched tubes. Finally, it remains to be demonstrated rigorously that no other shapes of horn lattices have harmonically related eigenfrequencies.

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